Student Paper Prize:

Numerical Investigation of Border Ice Failure

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ABSTRACT

Border ice commonly occurs in small rivers and brooks with rapid flow; it grows slowly until a complete ice cover is formed. Border ice may be subjected to rapid changes in water level leading to its failure.

An analytical approach has been developed for modelling the hydrodynamic-structural interaction of border ice and its failure. The analysis considers the possibility of failure of border ice due to rapid changes in water level. The modelling process assumes elastic behavior of the ice and is based on a finite element technique which has been used to develop the FEM1 Model presented herein. The analogy of a beam on an elastic foundation has been used in the formulation the FEM1 Model.

The paper investigates the possibility of failure of border ice with different variations in slope of the undersurface and hinged or fixed boundary conditions at the river bank, employing principal stress theory as a criterion for failure.

The results of this investigation are presented in a dimensionless form to enable examining the border ice for small rivers with different geometries and ice properties.

1. INTRODUCTION

The phenomenon of border ice formation in small rivers has been investigated numerically and experimentally by many researchers, Devik (1964), Hanley and Michel (1975 and 1977), Hirayama (1986), Matousek (1984), and Michel (1971). These investigations thoroughly examined air temperature, wind velocity, stream discharge, etc. and their effects on the rate of formation of border ice. The border ice may be subjected to rapid changes in water level. To the author's knowledge, these changes in water level and their effects on the cracking of the border ice, its failure and separation from the river bank have not been examined.

This paper presents a numerical model called FEM1 which has been developed based on the finite element technique. The paper investigates the hydrodynamic-structural interaction for border ice and its failure. The analogy of a beam on an elastic foundation has been used in the formulation of the FEM1 Model. The possibility of border ice failure is predicted on the basis of a combination of bending moments and shearing forces in the ice employing principal stress theory as a criterion for failure.

Many cases of border ice have been examined with different variations in slope of the undersurface, lengths, thicknesses, ice properties, and end conditions at the river bank. The variation in border ice geometries in different rivers and its properties in different cold regions at the time of the failure does not permit effective presentation of results for specific geometry and ice properties. Therefore, the model results have been generalized by presenting them in a dimensionless form. The solution stability and validity of the dimensionless results have been examined along with different combinations of input variables such as number of ice beam elements, thickness, length, uplift pressure, and elastic modulus. These variables have been used to obtain the relative stiffnesses and corresponding dimensionless deflections, bending moments, and shearing forces. It is submitted that the presented results will enable examining many small rivers with different widths, border ice geometry, and ice properties. Numerical examples are given at the end to demonstrate the use of the developed dimensionless formulation.

2. STATEMENT OF THE PROBLEM OF BORDER ICE

The border ice is initially buoyantly supported on the water and attached at the river bank. The application of the uplift pressure due to rapid changes in water level can be followed by cracking in the border ice and separation from the bank. The vertical structural equilibrium of an element of the border ice in such a case considers its own weight, the buoyant force, decrease on buoyant force due to deflection, and uplift pressure as shown in Fig. (1) and represented by the following equations:

$$P = p + \gamma\,(S_i\,H_i - Y) - \gamma_i\,H_i\,, \qquad (1)$$

or

$$P = p - \gamma\,Y. \qquad (2)$$

Here, P is net upward pressure, p is uplift pressure, γ, and γ_i are unit weights of water and ice, S_i is specific gravity of ice, Y, and H_i are the deflection and the border ice thickness.

The form of equation (2) implies that the uplift pressure is directly proportional to the deflection. This suggests that a unit length of the border ice in the direction of flow may be treated as a beam on an elastic foundation with modulus of foundation reaction equal to γ, the unit weight of water. Hetenyi (1946) examined the behaviour of a beam on an elastic foundation; he presented his solutions for the general differential equation governing the deflected shape of a uniform beam resting on a Winkler medium; such a medium was defined as one for which "the pressure in the foundation is proportional at every point to the deflection occurring at that point and independent of pressures or deflections elsewhere in the foundation". The differential equation can be written as follows:

$$EI\,\frac{d^4Y}{dX^4} + \gamma\,Y = p, \qquad (3)$$

or

$$EI\,\frac{d^4Y}{dX^4} + 4\,\lambda^4\,Y = \frac{p}{EI}\,. \qquad (4)$$

Here, $\lambda = \sqrt[4]{\gamma/4EI}$ is the characteristic of the system or the damping factor, EI is the flexural rigidity of the border ice for unit length, and X is the distance from the shore.

It should be noted that γ, the modulus of foundation reaction, is specified for the length border ice in the direction of flow. It will only be numerically equal to the unit weight of the water if the border ice is of unit length in the direction of flow which is true in this analysis in which $\gamma = 9.81$ kPa.

In Hetenyi's solution, it has been indicated that λ, the characteristic of the system, controls the distributions of deflection, rotation, bending moment, and shearing force for any particular case of loading.

Some researchers, Beltaos (1984 and 1985) and Billfalk (1981 and 1982) have employed Hetenyi's solutions for analysis of river ice covers. Their work considered a finite complete ice cover with symmetrical boundaries (hinged-hinged and fixed-fixed) and an infinite river ice cover. In both cases a uniform thickness for the ice cover was considered.

3. FEM1 MODEL FORMULATION AND COMPUTER IMPLEMENTATION

The formulation of the FEM1 Model for border ice is based on the finite element stiffness methodology for structures. In this method, the border ice is divided into a finite number of ice beam elements on an 'elastic foundation'. These elements are interconnected at nodal points which have a sequential numbering system to minimize the memory required during the computer calculations. It should be noted that it is important to have a considerable number of elements especially for varied thickness as is the case in border ice analysis.

All forces and displacements for an element of the border ice are shown in Fig. (2). Considering border ice with a finite number of beam elements, the external and internal forces can be related as follows:

$$\{P\} = [A]\ \{F\}; \qquad (5)$$

where {P} is an external forces vector, [A] is a statics matrix, and {F} is an internal forces vector.

Wang (1970) indicated, by using the reciprocal theory, that the internal and external displacements can be related as follows:

$$\{e\} = [A]^T\ \{X\}; \qquad (6)$$

where {e} is an internal deformations vector and {X} is an external displacements vector.

The internal forces can be related to the internal deformations as follows:

$$\{F\} = [S]\ \{e\}; \qquad (7)$$

where, [S] is a local stiffness matrix of the border ice.

Equilibrium, compatibility, and force-deformation requirements are satisfied by equation (5) through (7) which are the fundamental equations in the finite element analysis of border ice.

By substituting equation (6) into equation (7) the following equation can be written:

$$\{F\} = [SA^T]\ \{X\}; \qquad (8)$$

and substituting equation (8) into equation (5) the following equation can be written:

$$\{P\} = [ASA^T]\ \{X\} \qquad (9)$$

where, $[ASA^T]$ is the global stiffness matrix of the border ice.

Equation (9) represents a system of simultaneous equations which can be solved to obtain the displacements {X} after imposing boundary conditions. Then using equation (8), the forces induced in the border ice at the nodal points can be obtained. Imposing the fixed boundary conditions in the stiffness method has been made by considering very high stiffness for the rotational and vertical springs at the fixed boundary (i.e. no rotation and no vertical translation), and for the hinged boundary very high stiffness for the vertical spring and zero stiffness for the rotational spring (i.e. free to rotate and no vertical translation).

In the analogy of border ice with beam on an elastic foundation a provision has to be made to update the results especially for deflection. There is a subroutine named ICDISP in the FEM1 Model which solves the simultaneous equations and calculates the dimensionless deflections and rotations, and has a provision to stop the calculations if $Y_{max} > S_i H_i$ (the maximum allowable deflection) and continue if $Y_{max} \leq S_i H_i$. If FEM1 Model stops the calculations, it means that the free edge emerges from the water and the theory of beam on an elastic foundation is no longer valid. Also, to satisfy the analogy of a beam on an elastic foundation the border ice should not be flooded from the top.

The assumption of elastic response for the ice is valid for rapid changes in the water level as considered in this analysis. There is no universal agreement on the elastic limits for ice but some figures have been given by Sinha (1977), he suggested that for stresses less than 2 MPa and time less than 1 second or stress less than 0.5 MPa and time less than 100 seconds elastic behaviour can be assumed. Also, Ashton (1986) has reported limits in which the ice responds elastically to stresses less than 1 MPa and time less than 100 second or if ice is loaded to fail within 2 seconds based on Traetteberg et al. (1975).

4. DIMENSIONLESS DEFLECTION, BENDING MOMENT, AND SHEARING FORCE

Since the results are presented in a dimensionless form, it is found from equation (2) that a dimensionless group for the deflection can be written as follows:

$$\text{Dimensionless deflection} = \frac{\gamma Y}{p} . \qquad (10)$$

Equation (10) represents the proportionality between the deflection and pressure due to changes in water level.

The bending moment, M, and shearing force, Q, distributed along the border ice for ice length, L, are governed by the characteristic of the system, λ. If L, λ, M, Q, and p are considered variables for a dimensional analysis, the following dimensionless groups can be formed:

$$\text{Relative stiffness} = \lambda L; \qquad (11)$$

The characteristic of the system λ, is calculated based on the minimum thickness of the border ice at the free edge.

$$\text{Dimensionless shearing force} = \frac{\lambda^2 M}{p} , \qquad (12)$$

$$\text{Dimensionless shearing force} = \frac{\lambda Q}{p} . \qquad (13)$$

All results are presented in the next section referring to these dimensionless groups in equations (10) through (13). Also, the shore distance is considered in a dimensionless form (X/L).

5. RESULTS AND DISCUSSION

The dimensionless bending moment and deflections for border ice with uniform thickness and hinged ends at the river bank are presented in Fig. (3) and Fig. (4). In establishing these relationships and all others, different combinations of the input parameters such as Himin, E, L, and p, were used in obtaining different values of λL from 1 to 6. All dimensionless relationships have been checked by using different element sizes until a stable solution was obtained. Also, the solutions obtained by the FEM1 Model have been verified by comparing them with the available solutions by Hetenyi (1946) for uniform thickness. This confirmed the validity and the stability of the results. It can be seen that the position of the maximum dimensionless bending moment tends to move toward the shore with increasing relative stiffness λL (Fig. (3)). The maximum dimensionless deflection occurs at the free edge of the border ice for $\lambda L \leq 3$ as shown in Fig. (4). It tends to move toward the centre of the border ice for $\lambda L \geq 4$. The dimensionless bending moment for uniform thickness of border ice with fixed ends is shown in Fig. (5). It can be seen that the value of the maximum dimensionless bending moment always occurs close to the bank. The dimensionless deflection which tends to behave similar to the hinged end case is presented in Fig. (6).

In the case of variable thickness, four different slopes of the under surface (2.5°, 5°, 7.5°, and 10°) have been assumed. This range is assumed to cover most of the border ice formed in small rivers.

The dimensionless bending moments for the four slopes, a range of λL from 1 to 6, and hinged ends are shown in Fig. (7), Fig. (9), Fig. (11), and Fig. (13). These figures show an insignificant change in the dimensionless bending moment with increasing slope of the undersurface for each λL from 1 to 6. Such change vanishes for slopes greater than 5°. The corresponding dimensionless deflections are presented in Fig. (8), Fig. (10), Fig. (12), and Fig. (14). These figures show an insignificant change in the dimensionless deflection with increasing slope of the undersurface and the relative stiffness λL. Such change vanishes for slopes greater than 5°.

It should be pointed out that all dimensionless relationships for nonuniform thickness have been established based on the characteristic of the system at the free edge (i.e. at H_{imin}). Also, the border ice with slopes 2.5° and more has a large increase in flexural rigidity toward the bank for all relative stiffnesses; this explains the decrease in deflection and the behavior of border ice as a highly stiffened beam in all cases. Undersurface slopes from 0.5° to 2.0° are to be investigated for future applications of FEM1 Model to examine the changes in the dimensionless relationships, especially for deflections.

In case of border ice with uniform thickness, the predicted position of cracking is the same as the position of maximum dimensionless bending moment and maximum bending stress as well as the position of maximum principal stresses. The predicted position of cracking tends to move toward the bank with increasing the relative stiffness as shown in Fig. (15). However, for border ice with uniform thickness and a fixed end, the predicted position of cracking is always close to the bank which represents the position of maximum dimensionless bending moment, maximum bending stress, and maximum principal stress. Such cracking positions for both cases indicates that the shear stress has a negligiable effect.

The flexural rigidity was found to be the major factor affecting the predicted position of cracking for varied thickness of border ice with a hinged end. The flexural rigidity caused the predicted position of cracking to shift toward the free edge from the position of maximum dimensionless bending moment. However, the predicted positions of cracking, maximum bending stress, and maximum principal stress coincided. The predicted position of cracking tends to move toward the free edge with increasing slope of the undersurface for any given relative stiffness from 1 to 6 as shown in Fig. (15).

The predicted position of cracking for fixed end border ice with varied thickness is close to the bank. In this paper, due to the limitation of space. The results for only one slope (2.5°) are presented for three different relative stiffnesses from 1 to 3. The dimensionless bending moment for these cases is presented in Fig. (16). It shows that the maximum dimensionless bending moment is close to the bank. Furthermore, it was found that the predicted cracking, maximum bending stress, and maximum principal stress coincided and were close to the bank. Figure (17) shows the dimensionless deflection; from which it can be seen that the position of maximum dimensionless deflection occurs at the free edge of the border ice.

The present formulation is also applicable to the case of border ice subjected to a drop in water levels with the undersurface in contact with the water everywhere.

6. NUMERICAL EXAMPLES

Example (1): Examine the failure of uniform thickness border ice for $H_i = 0.16$ m, $L = 10.0$ m, $E = 4600$ MPa, $p = 0.5$ kN/m, and flexural strength = 700 kPa for (a) hinged end and (b) fixed end.

(a) Hinged end case

These data give $S_i H_i = 0.147$ m, $\lambda = 0.2$ hence $\lambda L = 2$. Using results in Fig. (4) gives $Y_{max} = 0.068$ m < 0.147 m, Fig. (15) gives $X_c/L = 0.3$. Also using results in Fig. (3). and Fig. (18). $M = 1.5$ kN.m, and $Q = -0.035$ kN. The principal stresses $\sigma_{1,2}$ can be obtained from

$$\sigma_{1,2} = 0.5\ \sigma_b \pm 0.5 \sqrt{\sigma_b^{\ 2} + 4\ \tau^2}$$

($\sigma_{1,2}$ are principal stresses, σ_b is bending stress, and τ is average shear stress). Hence, maximum principal tensile stress at top ≃ maximum bending stress at top = 353.5 kPa. Therefore, $p = 0.5$ kN/m is not suffient to cause cracking. However, if $p = 1.0$ kN/m by similar calculations it can be shown that cracking will occur at 3.0 m from the bank.

(b) Fixed end case

From the results in Fig. (6) gives $Y_{max} = 0.062$ m < 0.147 m, also using the results in Fig. (5) and Fig. (18) $M = -6.25$ kN.m, and $Q = -2.33$ kN.

Hence maximum principal tensile stress at bottom ≃ maximum bending stress at bottom = 1465 KPa. Therefore, $p = 0.5$ kN/m or less is enough to cause cracking close to the bank. This implies that the uplift pressure due to change in water level which is required to cause cracking for fixed end is less than that required for a hinged end.

Example (2): Examine the failure of border ice with variable thickness for $H_{imin} = 0.05$ m, $L = 4.0$ m, $\theta = 2.5°$, $E = 4600$ MPa, $p = 1$ kN/m, and flexural strength = 700 kPa for (a) hinged end and (b) fixed end.

(a) Hinged end case

These data give $S_i H_{imin} = 0.046$ m, $\lambda = 0.5$ hence $\lambda L = 2$. From the results in Fig. (8) $Y_{max} = 0.152$ m > 0.046 m and therefore the free edge emerges from the water level, and hence the methodology is not applicable.

(b) Fixed end case

From the results in Fig. (17) gives Ymax = 0.026 m < 0.046 m, also from Fig. (16), and Fig. (18) $M = -6.95$ kN.m, and $Q = -3.6$ kN. Hence, maximum principal tensile stress at bottom ≃ maximum bending stress at bottom = 830 KPa. Therefore, $p = 1.0$ kN/m which is sufficient to cause cracking close to the bank.

The limitation criterion built into FEM1 for the maximum deflection implies that for flooding from the top these calculations will be automatically stopped and so indicated in the output.

7. CONCLUSIONS

A FEM1 Model based on the finite element method coupled with the analogy of a beam on an elastic foundation has been developed to analyze the possibility of border ice cracking at the bank under rapid changes in water level.

It has been found that the predicted position of cracking for fixed end border ice with uniform and varied thickness of border ice is close to the bank. The hinged end border ice with uniform thickness has a predicted cracking position which tends to move toward the bank with increasing relative stiffnesses from 1 to 6. In these three cases the predicted position of cracking occurred at the position of maximum bending moment, which coincided with the location maximum bending stress and maximum principal stress. In case of variable thickness of border ice with hinged end, the predicted cracking position tends to move toward the free edge from the position of maximum bending moment with increasing slope of the undersurface for any given relative stiffness from 1 to 6. This displayed the importance of flexural rigidity. However the predicted cracking position coincided with the position of maximum bending stress and maximum principal stress in all cases. These results suggest that the shear stress has no effect on the predicted position of cracking. It was found that the changes in dimensionless bending moment for hinged border ice with varied thickness insignificant with increasing slopes between 2.5° to 10°, for each relative stiffness from 1 to 6. However, these changes vanish for slopes greater than 5°. It was also found that the changes in the corresponding dimensionless deflection are insignificant with increasing slopes and relative stiffnesses. Such changes vanish for slopes greater than 5°.

The numerical examples imply that the uplift pressures required to cause cracking in case of fixed end border ice are less than those required for a hinged end case with the same geometries, strength, and properties. This was found to be the case for border ice with uniform or variable thickness.

The developed dimensionless formulation can also apply for border ice subjected to a rapid drop in the water levels with the undersurface still in contact with the water everywhere. The border ice must be free from flooding at the top to satisfy the analogy of beam on an elastic foundation.

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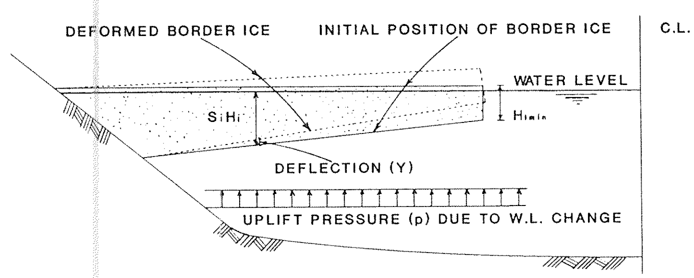

FIG. 1. DEFINITION SKETCH FOR THE PROBLEM

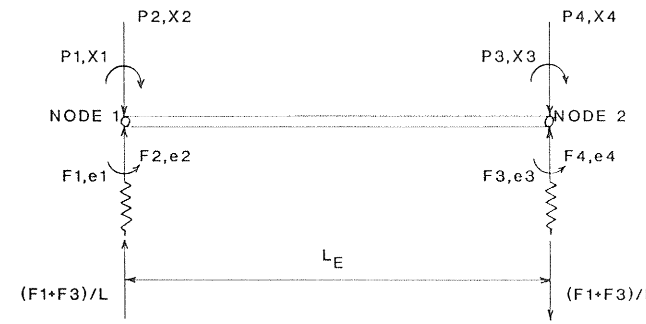

FIG. 2. IDEALIZATION FOR ICE BEAM ELEMENT (P & F FORCES AND X & e DISPLACEMENTS)

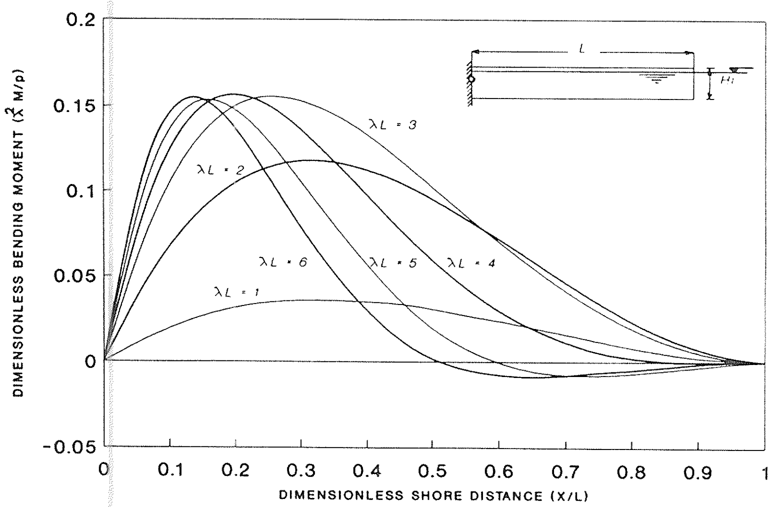

FIG. 3. DIMENSIONLESS BENDING MOMENT FOR HINGED BORDER ICE WITH UNIFORM THICKNESS

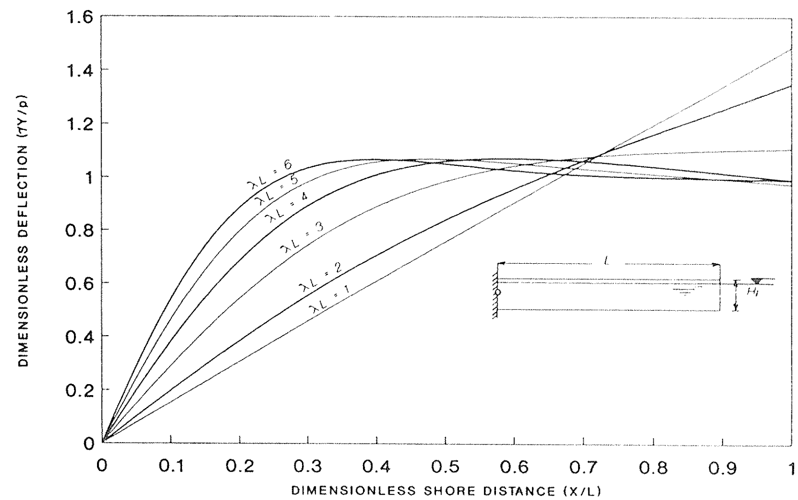

FIG. 4. DIMENSIONLESS DEFLECTION FOR HINGED BORDER ICE WITH UNIFORM THICKNESS

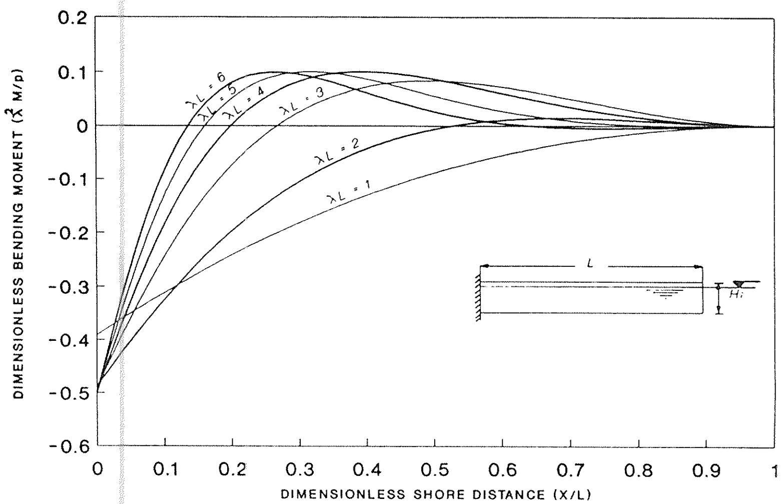

FIG. 5. DIMENSIONLESS BENDING MOMENT FOR FIXED BORDER ICE WITH UNIFORM THICKNESS

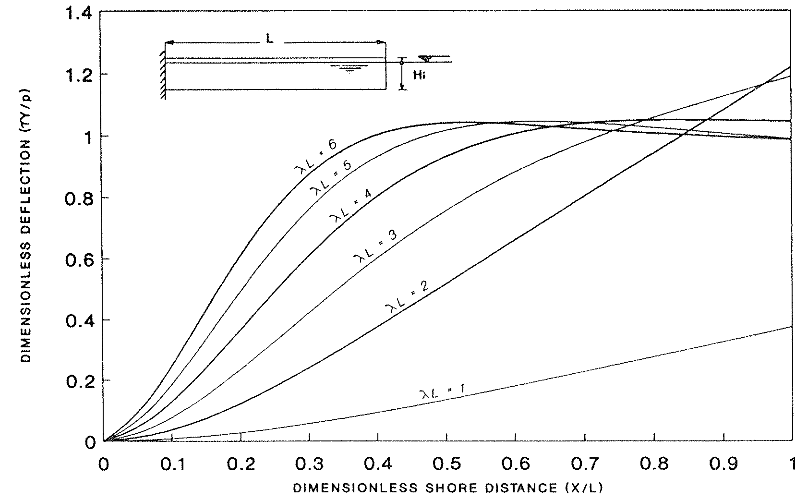

FIG. 6. DIMENSIONLESS DEFLECTION FOR FIXED BORDER ICE WITH UNIFORM THICKNESS

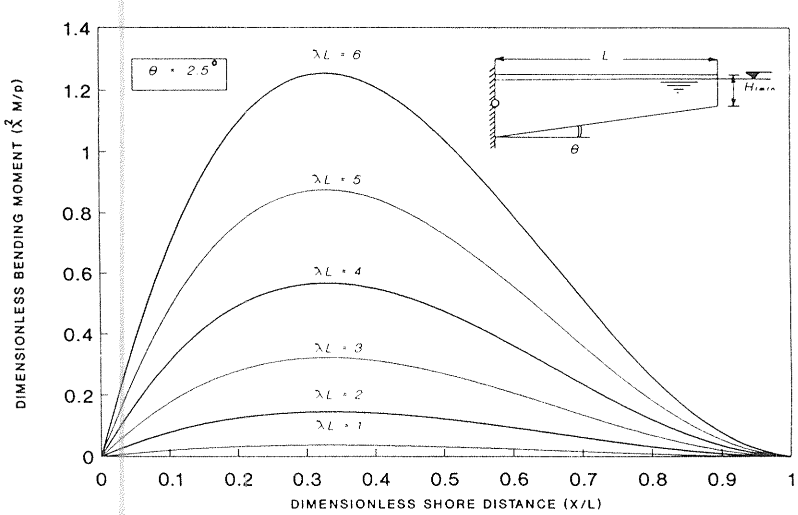

FIG. 7. DIMENSIONLESS BENDING MOMENT FOR HINGED BORDER ICE THICKENED UNIFORMLY

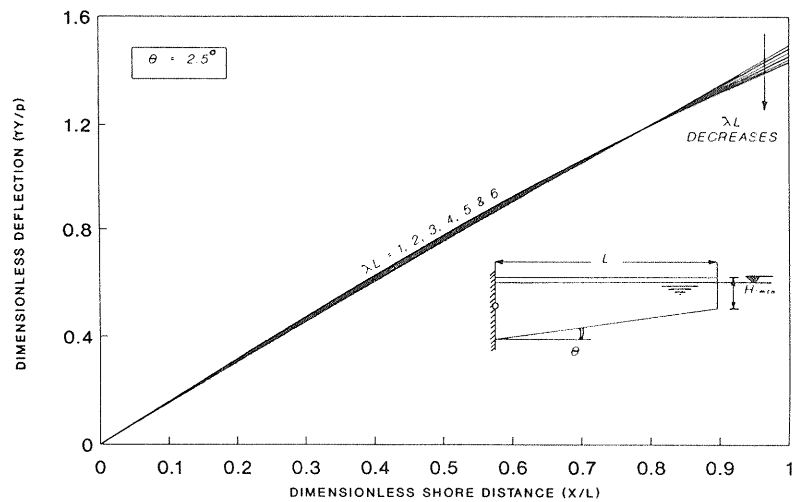

FIG. 8. DIMENSIONLESS DEFLECTION FOR HINGED BORDER ICE THICKENED UNIFORMLY

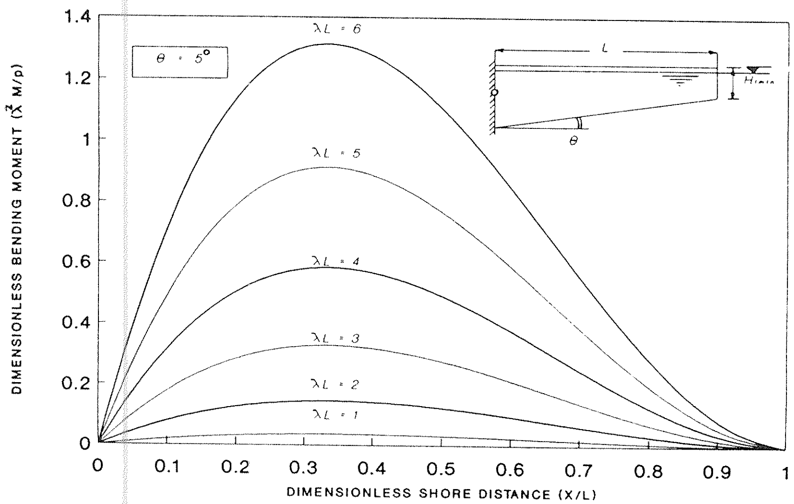

FIG. 9. DIMENSIONLESS BENDING MOMENT FOR HINGED BORDER ICE THICKENED UNIFORMLY

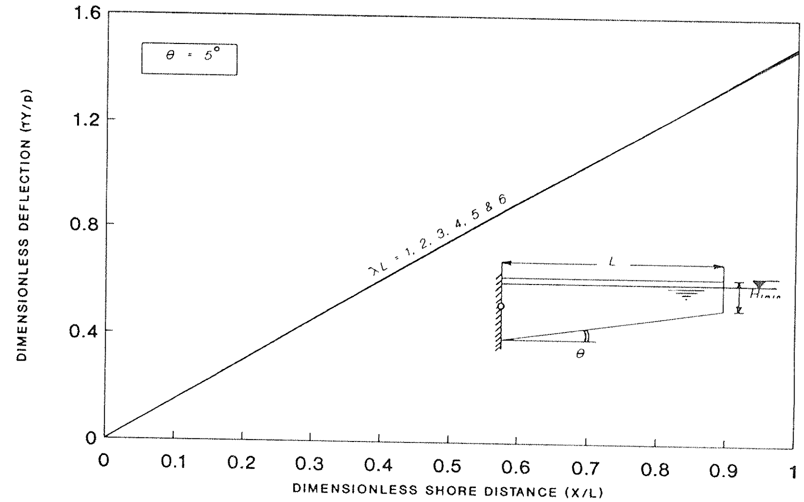

FIG. 10. DIMENSIONLESS DEFLECTION FOR HINGED BORDER ICE THICKENED UNIFORMLY

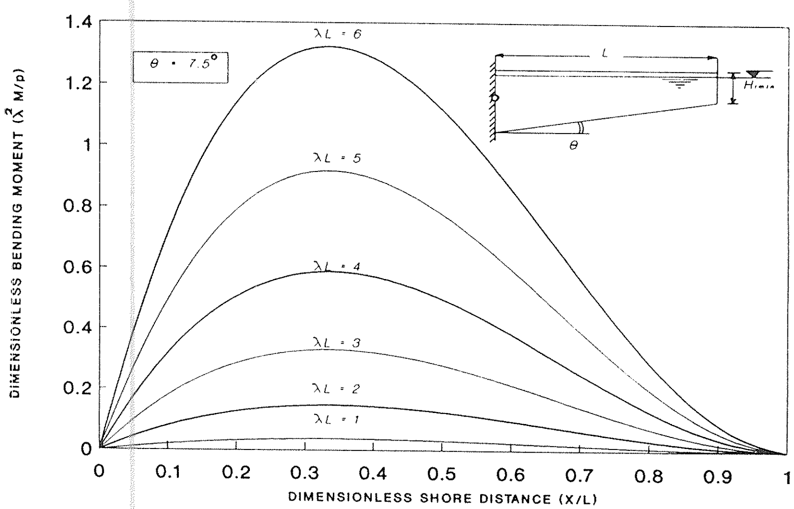

FIG. 11 DIMENSIONLESS BENDING MOMENT FOR HINGED BORDER ICE THICKENED UNIFORMLY

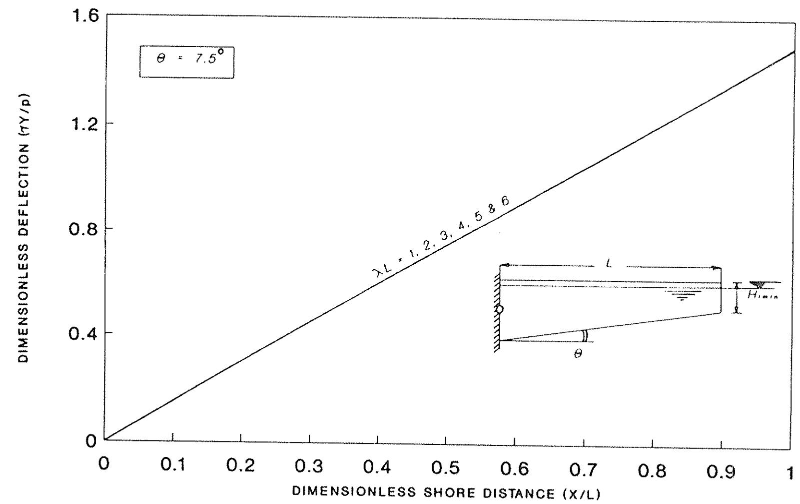

FIG. 12. DIMENSIONLESS DEFLECTION FOR HINGED BORDER ICE THICKENED UNIFORMLY

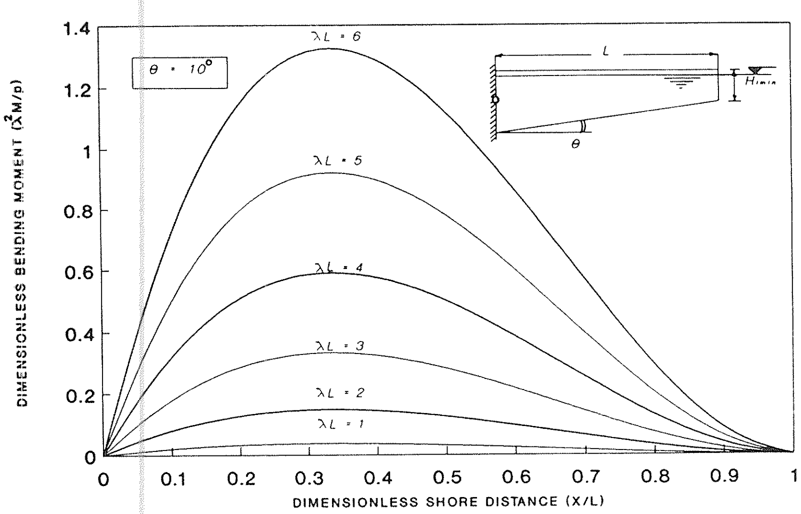

FIG. 13 DIMENSIONLESS BENDING MOMENT FOR HINGED BORDER ICE THICKENED UNIFORMLY

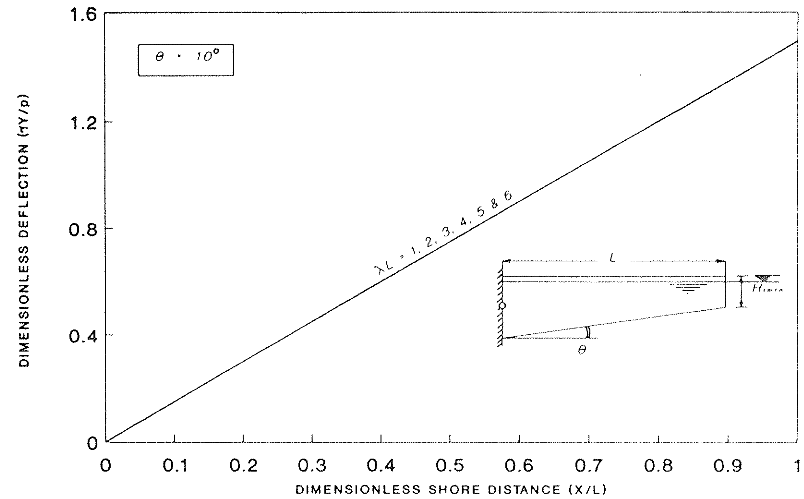

FIG. 14. DIMENSIONLESS DEFLECTION FOR HINGED BORDER ICE THICKENED UNIFORMLY

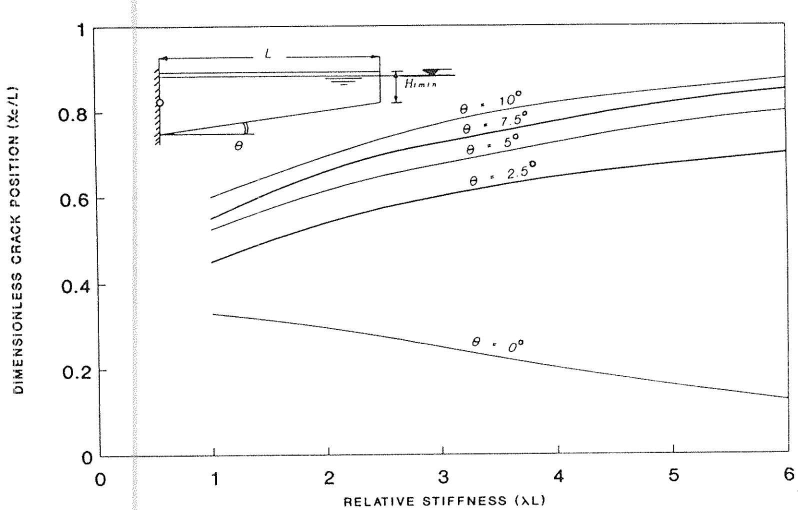

FIG. 15. DIMENSIONLESS PREDICTED CRACK POSITION FOR HINGED BORDER ICE

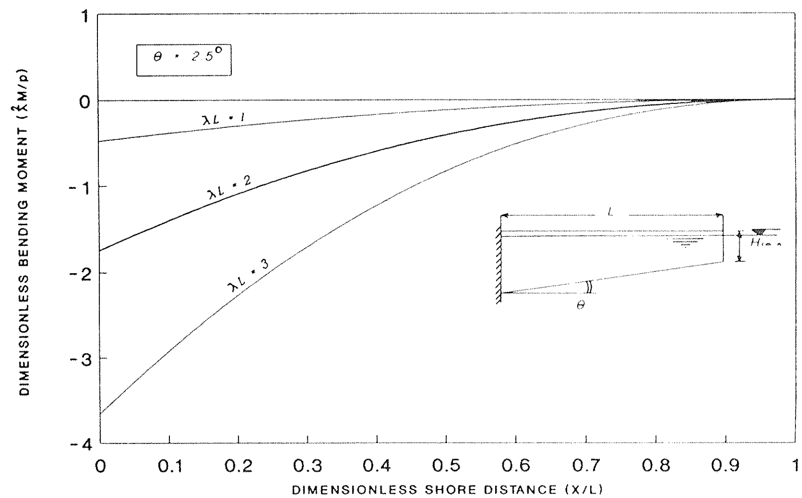

FIG. 16 DIMENSIONLESS BENDING MOMENT FOR FIXED BORDER ICE THICKENED UNIFORMLY

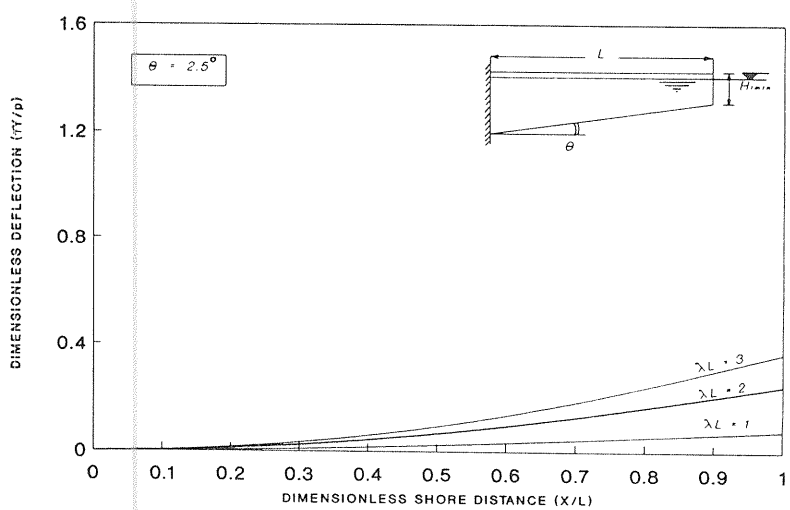

FIG. 17. DIMENSIONLESS DEFLECTION FOR FIXED BORDER ICE THICKENED UNIFORMLY

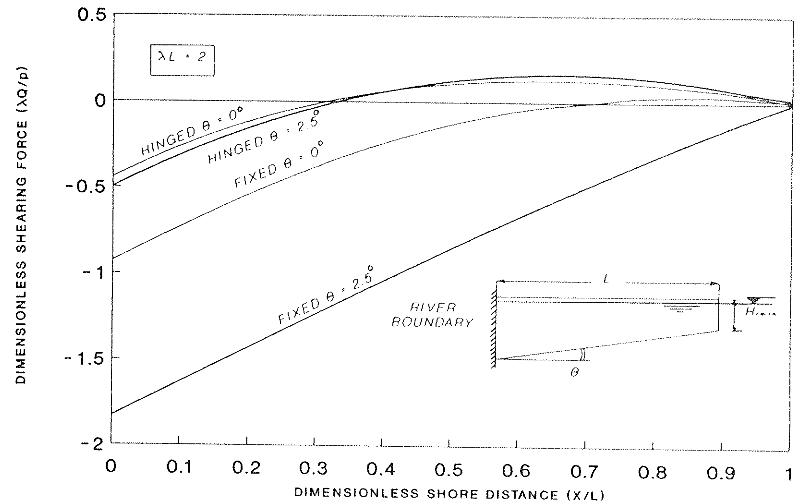

FIG. 18. DIMENSIONLESS SHEARING FORCE FOR BORDER ICE